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Cubical Homology and Robotic Motion Planning
方块同调和机器人路线规划

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Abstract

Homology is a theory used to study the topological spaces in algebraic topology. It can also be applied into the question of robotic motion planning. The study of the motion of automated robots has been an open area with numerous methods and challenges. The purpose of this thesis is to introduce some basic concepts about cubical homology and then try to use it to solve robotic motion planning question with the help of computer. Specifically, using homology group is a good way to exctract the information about the working loops of robots in some simple graphs.

Keywords: Cubical Homology, Discretized Configuration Space and Robotic Motion

同调论是代数拓扑里的一种方法，其常被用于研究拓扑空间。同调也可以被应用于机器人路径规划问题。关于自动机器人的运动问题的研究，一直以来是一个开放的领域充满了新方法和挑战。这篇文章的目的是介绍一些方块同调的基本概念，并且尝试通过电脑编程，解决此问题。具体来说，我们可以用同调，来提取有关于机器人在一些简单图上的工作回路的信息。

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Contents

Chapter 1

Introduction

1.1 Background and Motivation

With the increasing usage of robots in many industries, the question about robotic motion planning has attracted more and more attention in recent years. Consider the placement of several mobile robots, or automated guided vehicles(AGVs), controlled by a pre-defined algorithm in an automated factory. For example, commodities we bought online might be from a large warehouse, where all stocks are stored and stacked on shelves at very high level. Between those shelves are tracks that are connected with each other. Those robots are designed to transport items from place to place.

This goal needs to be compatible with two basical rules: functioning efficiently and avoiding collisions. Efficient performance involves two things. The first is the number of tracks between any two shelves should be as less as possible (only one track is what we will assume later), since more tracks means higher cost and less space for store. Secondly, if robots are able to finish the same amount of work with less or without repetitive or going-back movement, then it is considered as more energy-saving. With this understanding, it is natural to think if we can find a loop of motion, or cyclic paths for robots on a given track, in which they move as far as enough and then go back without repeating the tracks that has been visited.

The avoision of collisions means two or more moving robots are not allowed to function on a single track between two shelves. It might sounds easy that in reality, when shopping in a supermarket, we just need to swerve our charts slightly, if two charts are heading toward each other [1]. However, for automated robots, we assume they are not able to deal with this situation. Therefore, between any two shelves, only one robot are allowed to function at one time.

In this thesis, we consider a finite number of robots on a simple graph (both planar and non-planar). This might sound easy to see if we consider one or two robots by traditional methods like simulation. However, as the number of robots increase, the complexity of the problem will increase drastically. Imagining that two robots are working simultaneously on some simple graphs is already impossible for humans' brains.

In order to solve that, we record every state of the robotic motion system when they are working and correspond these states to points in configuration spaces. Basically, we are trying to convert the robotic information on graphs into the topological objects that are easier to deal with. Because a loop in the configuration spaces is nothing but a path with the same starting and ending point.

In order to find a loop, the idea of using homology then naturally comes out. As a powerful mathematical method for defining and categorizing holes in a topological spaces, it can also be applied into our question to detect loops in configuration spaces. It turns the seemly impossible question into a linear algebra question by computing the ranks of several matrices, which we will see in Chapter 2.

The rest of the thesis is organized as follows. In Section 1.2, we discuss some related literature and then we introduce some basic concepts about simplicial homology and manifold in Section 1.3. We discuss configuration spaces and cubical homology in Section 2.1, and we briefly show the used software and algorithm for modulo 2 Gaussian Elimination in Section 2.2. In Section 2.3, we give results of 2 robots' motion on some simple graphs. In Section 2.4, we introduce some possible future research. In Chapter 3, we discuss conclusions and future directions about the application of topology in robotic motion planning.

1.2 Literature Review

Homology, as a topic in algebraic topology, is about using the idea of quotient to extract information inside a topological space. It was originally utilized to analyse and characterize manifolds according to their loops. However, in recent years, more and more applications of homology have been intersected with other different fields such as data analysis, computational geometry and computer science, statistics and other areas. For instance, one method called persistent homology[2], has been used to detect qualitative features of data sets. It is appealing for its well-understood theoretical framework based on algebraic topology, computability by linear algebra and robustness about small perturbations in input data.

The theory of homology contains several types of complexes such as simplical complexes and cubical complexes. They are both useful. For simplicial complexes, roughly, they are spaces built from a union of vertices, edges, triangles, ployhedrons and higher dimensional polyhedrons. These structure can be used to extract information from a data set, such as the number of components and holes[2].

For a cubical complex, it is well-known for its application on digital images. Since two dimensional digital images are made of pixels and three dimensional images are made of voxels, digital images are endowed with a natural cubicl structure. Therefore, to study digital images, cubical complexes are more appropriate than simplcial complexes. Roughly, cubical complexes are made up of a union of vertices, edges, squares, cubes and so on, and then we glue them together with a specific rule. See [3] for an detailed discussion of cubical homology and its application on digital images.

Speaking of the problem of robotic motion planning on tracks, an detail discussion in [1] utilizes several simplification method: deformation-retracting a configuration space of a radial k-prong tree to get an explicit formula of its Euler characteristic; discretizing configuration spaces to count the number of faces, edges and vertices and then getting the number of genus by Euler characteristic. The second method shows its advantage with less complicated construction and without removing important information after discretization. In this thesis, we will follow this idea, use cubical homology to derive a computational method and then get the number of possible motion cycles.

1.3 Preliminary

1.3.1 Basic Homology

In this section, some basic mathematical definitions are introduced.

Definition 1.1 *Given a set* $\{p_0, ..., p_d\} \subset \mathbb{R}^N$*, the set is said to be* ***geometrically independent*** *if for any real scalars* α_i*, the equations* $\sum_{i=0}^{d} \alpha_i = 0$ *and* $\sum_{i=0}^{d} \alpha_i p_i = 0$ *imply that* $\alpha_i = 0$ *for all i. Given a geometrically independent set* $\{p_0, ..., p_d\}$*, a* ***d-simplex*** σ *spanned by* $\{p_0, ..., p_d\}$ *is the set of points of* $\mathbb{R}^N$*,* $\{\alpha_0 p_0 + \alpha_1 p_1 + ... + \alpha_d p_d \in \mathbb{R}^N | \sum_{i=0}^{d} \alpha_i = 1, \alpha_i \geq 0 \text{ for all } i\}$*. The number d above is called the dimension of* σ*. We say* σ' *is a* ***face*** *of* σ*, if it is a simplex spanned by a nonempty subset of* $\{p_0, ..., p_d\}$*.*

For example, 0-simplex is simply a point, 1-simplex is a line segment and 2-simplex is a triangle. Since $\{p_0, ..., p_d\}$ can *span* a simplex, henceforth, we use it to denote the d-simplex without causing confusion (The relationship between a simplex and its spanning set will be shown in the following definition of abstract simplicial complex).

Definition 1.2 *A **simplicial complex** K is a collection of simplices such that $\sigma \in K$ implies that any face of σ is also in K, and $\sigma_1, \sigma_2 \in K$ implies that $\sigma_1 \cap \sigma_2$ is either empty or a face of both. We say K is a simplicial complex of dimension d, if the largest dimension of all simplices in K is d. The **underlying space**, denoted by $|K|$, is the union of its simplices of K, with the topology defined by: given each simplex is closed (natural topology as as a subspace of $\mathbb{R}^N$), declaring a subset A of $|K|$ is closed in K if and only if $A \cap \sigma$ is closed in σ, for each simplex $\sigma \in K$.*

Definition 1.3 *A **subcomplex** of K is a simplicial complex $L \subseteq K$. The **j-skeleton** is a subcomplex $K^{(j)} = \{\sigma \in K | \dim \sigma \leq j\}$. The 0-skeleton is also called the **vertex set** of K.*

A useful subset of a simplicial complex is the star of a simplex. It will be used when we are discussing local neighborhoods in Section 1.3.2.

Definition 1.4 *Let τ be a simplex of K. The **star** of τ in K is St $v = \{\sigma \in K | \ \sigma$ is a face of $\tau\}$ Making it into a simplicial complex by adding all of its missing faces, then we call the simplicial complex, denoted by $\overline{St}$ v, the **closed star** of v in K. The **link** of τ in K is the union of all simplices in the closed star that are disjoint from τ, denoted by **Lk v**.*

If we consider a simplicial complex K of dimension 2, then the largest dimension of all simplices is 2. For a vertex $v = \{p_0\}$, its link is the union of all veticies that are joined by edges with $\{p_0\}$ and all edges that form triangles with $\{p_0\}$, i.e., $Lk\ \{p_0\} = \{\{p_i\} \in K | \ \{p_0, p_i\} \in K\} \cup \{\{p_j, p_k\} \in K \mid \{p_j, p_k, p_0\} \in K\}$. Here we use the spanning set to denote the corresponding simplex as mentioned before.

Definition 1.5 *A **triangulation** of a topological space X is a simplicial complex K together with a homeomorphism (a continuous and bijective function with continuous inverse) between X and $|K|$. A topological space is called triangulable if it has a triangulation.*

Without worrying about the topology, it is easier to consider a simplex with the finite vertex set that spans it. Thus, we will introduce the idea of abstract simplicial complex.

Definition 1.6 *An **abstract simplicial complex** S is a collection of finite non-empty sets, such that if $A \in S$, then every non-empty subest of A is an element of S.*

Given a simplicial complex, an abstract simplicial complex can be constructed from that simplical complex: the collection of subsets $\{a_0, a_1, ..., a_n\}$ of the vertex set of a simplicial complex, such that $a_0, a_1, ..., a_n$ spans a simplex in the simplicial complex. On the other side, given an abstract simplicial complex, a simplical complex is a geometrical realization of the abstract simplicial complex, in a space with sufficiently high dimension, see [4] for details. Now we are going to introduce homology for simplicial complex with modulo 2. Let $\mathbb{Z}_2$ be the field with two elements 0 and 1.

Definition 1.7 *Let K be a simplical complex, a **p-chain** is a finite sum of p-simplices $p = \sum c_i \sigma_i$, where each σ_i is a p-simplex in K and $c_i \in \mathbb{Z}_2$. The **p-chain group** $(C_p(K), +)$ is a group formed by the set of p-chains and the modulo-2 addition operator.*

The reason why we use $\mathbb{Z}_2$ is it has the simplicity of computation and potential geometric meaning, which will be shown later in the property of boundary operater. And note that $C_p(K)$ is actually a vector space over the field $\mathbb{Z}_2$ with the set of p-simplices as a basis.

Definition 1.8 *For every $p \in \mathbb{N}$, the **p-th boundary operator** is a linear map $d_p : C_p(K) \rightarrow C_{p-1}(K)$,*

$$\sigma \mapsto \sum_{\tau \subset \sigma, \tau \in K_{p-1}} \tau = \sum_{i=0}^{p} \{v_0, v_1, .., \hat{v}_i ..., v_p\}, \tag{1.1}$$

where σ is spanned by $\{v_0, v_1, ..., v_p\}$ and $\{v_0, v_1, ..., \hat{v}_i, ..., v_p\}$ is a face of σ that spanned by all other vertices except $\hat{v}_i$. And d_0 is the zero map.

Not hard to think that every (p-1)-face of a (p+1)-simplex is shared by two p-simplices as faces of the (p+1)-simplex. Consequently, two consecutive operators will map a simplex into 0 since we are doing addition in $\mathbb{Z}_2$. Then one of the most important properties of the boundary operator is the following:

Lemma 1.1 *The composition of two consecutive boundary operator is a zero map, i.e.* $d_{p+1}d_p = 0$.

Proof: Just need consider an (p+1)-dimensional simplex $\sigma = \{v_0, v_1, ..., v_{p+1}\}$. *Then*

$$\begin{aligned} d_p d_{p+1}(\sigma) &= d_p(\sum_{i=0}^{p+1}\{v_0, v_1, .., \hat{v_i}..., v_{p+1}\}) \\ &\overset{(1)}{=} \sum_{i=0}^{p+1} \partial_p(\{v_0, v_1, .., \hat{v_i}..., v_{p+1}\}) \\ &\overset{(1)}{=} \sum_{i=0}^{p+1}\sum_{j=0}^{i-1}\{v_0, ..., \hat{v_j}, ..., \hat{v_i}..., v_{p+1}\} \\ &+ \sum_{i=0}^{p+1}\sum_{j=i+1}^{p+1}\{v_0, ..., \hat{v_i}..., \hat{v_j}..., v_{p+1}\} \quad \overset{(2)}{=} 0, \end{aligned} \tag{1.2}$$

where (1) comes from ∂ *is linear and (2) comes from the modulo 2 addition.* ■

It shows that image(d_{p+1}) $\subseteq$ kernel(d_p). As a result, we can define the two subgroups of a chain group and their quotient group.

Definition 1.9 *The* ***p-cycle group*** Z_p *is the collection of p-chains* γ *that satisfies* $d_p(\gamma) = 0$. *The* ***p-boundary group*** B_p *is the collection of p-chains* γ *that satisfies that it is a boundary of a (p+1)-chain, i.e,* $\exists\delta$, *such that* $d_{p+1}(\delta) = \gamma$. *The* ***p-homology group*** *is the quotient group of the above two* $H_p = Z_p/B_p$. *The* ***p-th Betti number*** *is* $\beta_p = log(card(H_p)) = log(card(Z_p)/card(B_p))$, *where card is the cardinality of a group.*

Homology group indicates that if two p-cycles are differenced by the boundaries of several simplices, they are intrinsically equal. Furthermore, for p larger than zero, a p-cycle in a p-homology group can be viewed as a p-dimensional hole that lies in the space and the p-th Betti number provides a method for us to count the number of holes. In particular, $dim(H_0(X))$, $dim(H_1(X))$ and $dim(H_2(X))$ are the number of connected components, 2-dimensional holes and 3-dimensional voids respectively.

1.3.2 Manifold

Manifold as a core subject in topology will be introduced briefly here.

Definition 1.10 *An* ***n-manifold*** *is a topological space M such that each point of it has a neighborhood D that is homeomorphic (exists a continuous and bijective function with continuous inverse, from D to* $\mathbb{R}^n$*) to the Euclidean space of dimension n; i.e., it can be covered by open sets (charts) homeomorphic to* $\mathbb{R}^n$.

Intuitively, if M is a 2-manifold (also called surface), then it means that M looks locally like the plane.

Next we will introduce a useful method to analyze manifolds: **Triangulation**. A theorem proved by Whitehead in [5], states that every compact 2-manifold is triangulable. A triangulation K of a 2-manifold M is automatically a simplicial 2-complex consisting of vertices, edges and triangles. We may orient each triangle to define the orientation of manifolds. Two triangles sharing an edge are **consistently oriented** if they induce opposite orientations on the shared edge. Then the manifold M is **orientable** iff the triangles can be oriented in such a way that every adjacent pair is consistently oriented.

We now only consider a triangulation with finite simplices of a 2-manifold. Letting n, m, and k be the numbers of vertices, edges, and triangles in the triangulation, the **Euler characteristic** χ is their alternating sum, $\chi = n - m + k$. A surface is **closed**, if it is compact, connected, and has no boundary. By the **classification theorem for closed 2-manifolds**, closed 2-manifolds, with different numbers of orientable holes and non-orientable cross-caps, exhaust all of them. Moreover, they are determined by the Euler Characteristic, up to homeomorphism.

In terms of homology, the Euler Characteristic can also be obtained by the alternating sum of Betti numbers $\chi = \beta_0 - \beta_1 + \beta_2$, by **Euler-Poincare Theorem**. Thus, it can be said that Betti numbers are a refinement of the Euler characteristic, as they involves more information of the topological space. More importantly, homology groups and Euler characteristic are both intrinsic information of a topological space, which do not depend on the triangulation chosen.

In fact, it also provides a nice method to tell two 2-manifolds apart. If we have two 2-manifolds, checking if their topological invariants (such as their homology groups) are the same is a great way to distinguish them without considering their geometry. Given the two triangulations, computing their homology groups and Betti numbers is straightforward. Furthermore, we can see the orientability with the integer coefficient homology: $\beta_2 = 0$ for an non-orientable closed 2-manifold; $\beta_2 = 1$ for an orientable one.

Next, we will give a necessary and sufficient statement of a 2-manifold. If a topological space is a closed 2-manifold, then in any triangulation of that manifold, we have: (1) the link of each vertex is a closed path made up of edges and vertices in K, which is homeomorphic to a circle or a 1-dimensional sphere; (2) the link of each edge is the set of two points, which is homeomorphic to a 0-dimensional sphere. For the necessary direction, an triangulation must exist as mentioned before and it is easy to demonstrate that by checking the link of every edge and vertex of that triangulation. For the sufficient direction, if a point lies at a vertex, then that the link of that vertex is a cycle is equivalent to it is surrounded by at least 3 triangles, which means a neighborhood of that point is homeomorphic to $\mathbb{R}^2$. If a point lies on an edge, then that the link of the edge is two point is equivalent to it borders on a pair of two triangles, which means the point has a neighborhood homeomorphic to $\mathbb{R}^2$. Finally, if a point lies in the interior of a triangle, it is trivial.

Chapter 2

Robotic Planning

2.1 Configuration Spaces

This chapter will outline ideas about configuration spaces and the application of homology group on them.

Definition 2.1 *The **configuration space** of n distinct labeled points (or robots) on a topological space X, denoted by $C^n(X)$, is the space*

$$C^n(X) = \prod_{1}^{n} X - \Delta, \tag{2.1}$$

where Δ denotes the diagonal

$$\Delta = \{(x_1, ..., x_n) : x_i = x_j,\ for\ some\ i \neq j\}. \tag{2.2}$$

Given a topological space X, any point in the configuration space is a state of the system of the n robots. The diagonal here means, to navigate safely on the graph, impending collision between robots must be avoided. In other words, paths on $C^n(X)$ should leave the diagonal Δ alone.

We mainly focus on the motion of n robots on a simple connected graph Γ. In other words, a loop with the same ending point and starting point does not exist; two parallel edges incident on two same vertices do not exist. Since a simple graph consists of vertices (0-cubes) and edges (1-cubes), it naturally holds a cubical structure. As a result, the n-fold Cartesian product of Γ also persists the structure. Actually, it forms a cubical complex, which is a cell complex built from finite-dimensional Euclidean cubes inductively. Then we glue or identify them together along faces by gluing maps and finally take quotient. However, we will omit the complicated process to simplify our problem without causing confusion. Because if we label every vertex of a graph Γ by distinct natural numbers, then gluing along faces is automatic.

Definition 2.2 *An **elementary interval** in $\mathbb{R}$ is $I = [l, l+1]$ or $I = \{l\}$ (degenerate interval) for some $l \in \mathbb{Z}$. An **elementary cube** $Q = I_1 \times I_2 \times ... \times I_n \subseteq \mathbb{R}^n$ is a Cartesian product of elementary intervals. We also define the **embedding number** of Q is $\mathrm{emb}(Q) = n$ and the **dimension** of Q is $\dim(Q) = k$, if the number of non-degenerate intervals in the product is k. We call Q an elementary k-cube, if its dimension is k. The set of all elementary cubes in $\mathbb{R}^n$ is denoted by $\mathscr{C}^n$. The set of all elementary cubes is denoted by $\mathscr{C}$. The set of all elementary cubes of dimension k in $\mathbb{R}^n$, is denoted by $\mathscr{C}^n_k = \{Q \in \mathscr{C}^n |\ \dim(Q) = k\}$. A **face** of Q is a subset of Q as an elementary cube.*

The embedding number here looks like redundant, but it will be utilized in the following part of boundary operators. Compared to the case of simplicial complex where we are able to use a finite vertex set to represent a simplex, it is truly something we need to fuss with.

Definition 2.3 *A set $X \subseteq \mathbb{R}^n$ is* ***cubical*** *if it is a finite union of elementary cubes of embedding number n. We also use the following notation $\mathscr{C}(X) := \{Q \in \mathscr{C}^n | \ Q \subset X\}$ to define the set of elementary cubes of X, and $\mathscr{C}_k(X) := \{Q \in \mathscr{C}(X) | \ \dim(Q) = k\}$ to define the set of elementary cubes of dimension k in X.*

Note that in the definition of a cubical set, we may omit the superscript(embedding number) n, since the embedding number is already given. We take $X = [1,2] \times \{1\} \times [0,1]$ as an example. This is an elementary cube and also a cubical set. Now, we list the the set of elementary cubes of X at different dimensions: $\mathscr{C}_0(X) = \{\{1\} \times \{1\} \times \{0\}, \{2\} \times \{1\} \times \{0\}, \{2\} \times \{1\} \times \{0\}, \{2\} \times \{1\} \times \{1\}\}$, $\mathscr{C}_1(X) = \{[1,2] \times \{1\} \times \{0\}, [1,2] \times \{1\} \times \{1\}, \{1\} \times \{1\} \times [0,1], \{2\} \times \{1\} \times [0,1]\}$, and $\mathscr{C}_2(X) = \{[1,2] \times \{1\} \times [0,1]\}$.

The reason why we introduce the idea of cubical set X is because it is a union of elementary cubes, which is the topological space as a whole we want to analyze after by the theory of homology. It is also an analogy to the underlying space of a simplicial complex as mentioned before. In contrast, $\mathscr{C}^n(X)$ is a collection of elementary cubes, by which we are able to deal with the elementary cubes individually, as shown in the following definition of boundary operators.

We then consider n robots moving along edges and vertices of an undirected simple graph. For such a graph $\Gamma = (V, E)$, we index the vertices with natural numbers: $V = \{\{1\}, ..., \{m\}\}$ where m is the number of vertices. Then we label edges by the unordered pair of vertices $\{1, 2\}$ to indicate the edge incident on the two endpoints $\{1\}$ and $\{2\}$.

Note that by a straight-line embedding of the graph Γ to $\mathbb{R}^2$, every edge or vertex is homeomorphic to an elementary interval. Since that, we still use the notations of edges and vertices, but see them as elementary cubes without causing confusion. Analogous to the definition of cubical set, we also transform the graph into a topological space. Namely, we define the finite union of elementary cubes consisting of edges as: $\widetilde{\Gamma} = \cup(\{\{l, l'\} | \{l\}, \{l'\} \in V \text{ and } l < l'\})$. We ignore vertices, since they are already in edges when we take the union. Therefore, both $\widetilde{\Gamma}$ and the Cartesian n-fold product of $\widetilde{\Gamma}$ are cubical sets.

Definition 2.4 *Let $\Gamma = (V, E)$ be a simple undirected graph labeled by natural number as $V = \{\{1\}, ..., \{m\}\}$, where m is the number of its vertices and $E = \{\{l, l'\} | \{l\}, \{l'\} \in V \text{ and } l < l'\}$. $\widetilde{\Gamma}$ is defined as above.*

The ***discretized configuration space*** *of $\widetilde{\Gamma}$ is*

$$D^n(\widetilde{\Gamma}) = \prod_{1}^{n} \widetilde{\Gamma} - \widetilde{\Delta}, \tag{2.3}$$

where $\widetilde{\Delta}$ is the set of elementary cubes that intersect with diagonal Δ.

The motivation of introducing the concept of the discretized configuration space is to find a cubical set. Because, after removing the diagonal Δ, an annoying problem comes out: the configuration space is neither compact nor holds a cubical structure any more. The diagonal cuts through all product cubes with pairwise repeated elements. To solve that, we approximate configuration spaces of graphs by discretization. After removing all elementary cubes in $\prod_1^n \widetilde{\Gamma}$ that intersect with the diagonal Δ, the expected cubical structure is still preseved. Thus, $D^n(\widetilde{\Gamma})$ is a cubical set. Described on the graph Γ, any path in Γ that connects any two robots should satisfy: the path contains at least one entire edge. This discretization will simplify our problem and this simplification will be seen to bring some good mathematical results. It also helps transform the information of a graph into a cubical set, which can be analyzed by a computer.

Again, analogous to simplical complex, we are still talking about homology in the context of modulo 2 coefficient.

Definition 2.5 *A* ***cubical k-chain*** *c of a cubical set X is a finite sum of elementary cubes $Q_i \in \mathscr{C}_k(X)$, $c = \sum \alpha_i Q_i$, with the coefficients $\alpha_i \in \mathbb{Z}_2$. The addition operation of two cubical k-chains*

$c_1 = \sum \alpha_i Q_i$ and $c_2 = \sum \beta_i Q_i$ is defined to be $c_1 + c_2 = \sum(\alpha_i + \beta_i)Q_i$. The ***cubical k-chain group*** *$(C_k^n(X), +) = C_k^n(X)$ is a abelian group of the set of all cubical k-chains in the set X, equipped with the addition mentioned above.*

Definition 2.6 *Let X be a cubical set. For every $k \in \mathbb{N}$, the* ***k-th cubical boundary operator*** *is a homomorphism $\partial_k : C_k^n(X) \to C_{k-1}^n(X)$, defined by induction on the embedding number n as the following (One can omit the subscript k, since it is automatically determined by the domain of the k-th boundary map):*

First of all, the case $n = 1$. ∂ is defined as the map, for any elementary cube $Q \in \mathscr{C}^1(X)$,

$$\partial(Q) = \begin{cases} 0, & \text{if } Q = \{l\} \in \mathscr{C}_0^1(X) \\ \{l\} + \{l'\}, & \text{if } Q = \{l, l'\} \in \mathscr{C}_1^1(X). \end{cases} \tag{2.4}$$

Then assume $n > 1$. For any elementary cube $Q = I_1(Q) \times \cdots \times I_n(Q) := I_1(Q) \times P$, $P \in \mathscr{C}^{n-1}(X)$,

$$\partial(Q) = \partial I_1(Q) \times P + I_1(Q) \times \partial P. \tag{2.5}$$

Also, we define ∂_0 is the zero map. Finally, we get the boundary map for a cubical chain $c = \sum \alpha_i Q_i$ by its linearity.

Proposition 2.1 $\partial\partial = 0$.

Proof: We only need to show it holds for any elementary cube Q and it will be proved by induction on the embedding number n.

First, consider the case $n = 1$. If $Q \in \mathscr{C}^1$, then $\partial\partial Q = 0$ by the definition.

Then assume the property holds for $Q \in \mathscr{C}^n$, $n = m > 1$. Then need to show it also holds for $m + 1$:

$$\begin{aligned} \partial\partial(Q) &= \partial\partial I_1(Q) \times P + \partial I_1(Q) \times \partial P + \partial I_1(Q) \times \partial P + I_1(Q) \times \partial\partial P \\ &\overset{(1)}{=} \partial\partial I_1(Q) \times P + I_1(Q) \times \partial\partial P \\ &\overset{(2)}{=} I_1(Q) \times \partial\partial P \\ &\overset{(3)}{=} 0 \end{aligned} \tag{2.6}$$

The equality (1) comes from that we are doing addition in the field $\mathbb{Z}_2$. The equality (2) comes from the case $n = 1$. The equality (3) comes from our assumption. ■

Definition 2.7 *Let X be a cubical set. The* ***cubical k-cycle group*** *$Z_k(X)$ is the collection of cubical k-chains $\gamma \in C_k(X)$ that satisfies $\partial_k(\gamma) = 0$. The* ***cubical k-boundary group*** *$B_k(X)$ is the collection of cubical k-chains $\gamma \in C_k(X)$ that satisfies that it is a boundary of a cubical (k+1)-chain, i.e, $\exists \delta$, such that $\partial_{k+1}(\delta) = \gamma$. The* ***cubical k-homology group*** *is the quotient group of the above two $H_k = Z_k(X)/B_k(X)$. The* ***cubical k-th Betti number*** *is $\beta_k = log(card(H_k)) = log(card(Z_k)/card(B_k))$, where card is the cardinality of a group.*

From Proposition 3.1, it is easy to see that $B_k(X) \subseteq Z_k(X)$ and $B_k(X)$ is a normal subgroup since they are all abelian. It is also worth to mention that $C_k(X)$ over the field F_2 is a vector space and the set $\mathscr{C}_k(X)$ is its basis. $B_k(X)$ and $Z_k(X)$ are also vector spaces with their corresponding basses. Therefore, $\beta_k = \dim(Z_k(X)) - \dim(B_k(X))$.

Example We consider two robots' motion on the graph Γ consisting of four edges of a square. We index four vertices as $V = \{\{1\}, \{2\}, \{3\}, \{4\}\}$, $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{1,4\}\}$ and $\widetilde{\Gamma} = V \cup E := \square$. We will also simplify the notation of elementary cubes $I_1 \times I_2$ by removing their curly brackets and replacing $\times$ by $|$. For instance, $1|23$ represents the elementary 1-cube $\{1\} \times \{2, 3\}$ or the state that the first robot stays on vertex $\{1\}$ and the second is at some point of the edge $\{2, 3\}$.

$D^2(\square)$ now is a cubical set as mentioned before. Each vertex (0-cube) in $D^2(\square)$ corresponds to a state where two robots are at distinct vertices of $\square$. Each edge (1-cube) in $D^2(\square)$ corresponds to a state where one of the two robots is at a vertex of $\square$, and the other is on an edge with different endpoints. Each 2-cube in $D^2(\square)$ corresponds to a state where two robots are at distinct edges of $\square$, whose four endpoints are all different.

Speaking of the diagonal, the discretized configuration space $D^2(\square)$ is a proper subset of $C^2(\square)$, as it removes more parts than $C^2(\square)$. Here, $\Delta = \{(x_1, x_2) : x_1 = x_2\}$ is a line and $\widetilde{\Delta} = \{Q \in \mathscr{C}^2 : Q \cap \Delta \neq \emptyset\}$. That is, we need to remove the four elementary 2-cubes that intersect with the diagonal and all of its faces, except the eight elementary 0-cubes that do not interest with the diagonal. For instance, remove 12|12 and all faces inside but keep 1|2 and 2|1. Illustrated in the graph, if the first robot is at vertex 1, then the other cannot stay at vertex 1 or edge 12 or edge 14. In other words, every pair of robots must stay at least a edge apart from each other.

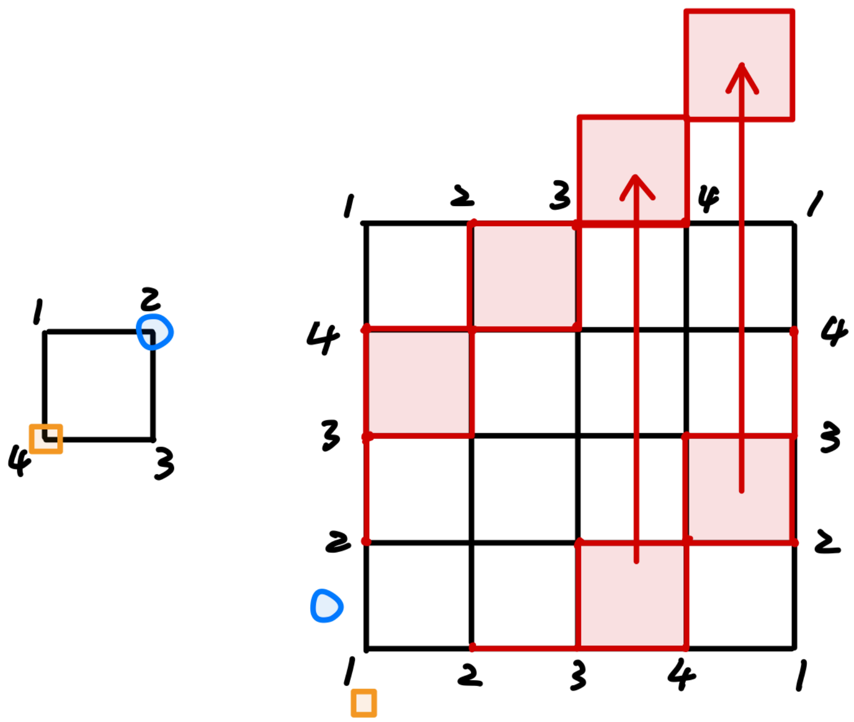

Figure 2.1: An example of 2 robots' motion on four edges of a squre

In Fig. 2.1, we show the visualization: the discretized configuration space of $D^2(\square)$ consists of marked red vertices, edges and cubes. Moving the bottom upwards and identifying edges and vertices with the same labelling gives a clear picture. It forms a 'necklace' with four 'pearls'.

We also give an example of the cubical boundary operator, $\partial_2(12|34) = 1|34 + 2|34 + 12|3 + 12|4$. With a little calculation, we get the cubical 0-th Betti number $\beta_0 = 1$ (one connected component) and the cubical first Betti number $\beta_1 = 1$ (one 2-dimensional holes). The cubical first Betti number also counts the number of working loops of the two robots on the graph $\square$ with modulo 'boundary'. That is, given a working loop, any other working loop can be obtained by adding some boundaries of 2-cubes to the pre-specified one. Also note that, the two robots rotate around the square once with the counterclockwise and clockwise directions are treated equally here. It provides us a good measurement of how many working cycles inside without struggling to tell similar ones apart.

2.2 Algorithm

However, it is not easy to compute the cubical Betti number for $D^n(\Gamma)$, if n is bigger than 2 and graph Γ is more complicated than $\square$. Because, we have to check if a chain is a cycle, and then find all boundaries, and finally take quotient. What is worse, at most time, the configuration space is not a manifold, which means characterization theorem cannot be applied here. However, since they are all finite dimensional vector spaces, we are able to use matrix for help. In this case, we turn to use computer to do the job.

Boundary Matrix Let X be a cubical set. Its k-th boundary matrix represents the elementary cubes of dimension (k-1) (or elementary (k-1)-cube) as rows and the elementary cubes of dimension k as columns, by assuming an arbitrary but a fixed ordering of the elementary cubes of dimension k and then listing their faces at dimension (k-1). This matrix is $\partial_k = [a_{i,j}]$, where i ranges from 1 to n_{k-1}, j ranges from 1 to n_k, where n_k is the number of elementary cubes of dimension k in X.

$$a_{i,j} = \begin{cases} 1, & \text{if the i-th elementary (k-1)-cube is a face of the j-th elementary k-cube} \\ 0, & \text{otherwise} \end{cases}$$

$$\partial_k = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n_k} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n_k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_{k-1},1} & a_{n_{k-1},2} & \cdots & a_{n_{k-1},n_k} \end{pmatrix}$$

Also recall the fact that for any linear transformation between vector spaces $f : U \to V$, the property $\dim(U) = \dim(\text{Kernel(f)}) + \dim(\text{Image(f)})$ holds. As a result, writing $z_k = \dim(Z_k) = \dim(\text{Kernel}(\partial_\text{k}))$ and $b_k = \dim(B_k) = \dim(\text{Image}(\partial_\text{k})) = rank(\partial_k)$, this can be stated as $n_k = z_k + b_{k-1}$ and then $\beta_k = z_k - b_k$ is easy to compute.

By Gaussian Elimination, we compute the rank of a boundary matrix in modulo 2. In fact, in algorithm 1, we are operating on the transpose of a boundary matrix. It has some clear algebraic meanings: after transposing, we do elementary row operations to eliminate 1s to 0s, which corresponds to find a linear combination of rows (elementary k-cubes) that equals to zero, which is exactly a cubical k-cycle.

Need to note that, if we do mod 2 operation after all elementary row operation is finished, the transformed row echelon form is in fact an approximated one. Because in computer, if we want to eliminate the first non-zero elements of some rows by addition, it will involve some fraction numbers and they will be approximated by close decimal numbers. Thus, every time after row addition, we should follow by mod 2 operation to avoid that. See algorithm 1 for details.

Software Using **DataFrame** of Pandas in Python, it is clear to see the boundary matrix and then performing elimination with its built-in functions. See Fig. 2.2 for the first cubical boundary operator ∂_1 of $D^2(\square)$ as an example.

2.3 Results

This section will show some results of the cubical Betti numbers of discretized configuration spaces of two robots.

2.3.1 Radial k-Prong Trees

This subsection will show results of two robots on some radial k-prong trees. For each $k > 2$, a radial k-prong tree T_k is a tree with vertices $\{v_i\}_0^k$ and edges $\{e_i\}_1^k$ attaching the central vertex v_0 to the outer vertices $\{v_i\}_1^k$. See examples in 2.3 and 2.4.

As stated in [1], there is an explicit formula for Euler characteristic of the configuration space of n robots on a radial k-prong tree $C^n(T_k)$:

Algorithm 1 Mod 2 Gaussian Elimination Algorithm

```
function M2Elimination(A)
    m = the number of rows of matrix A
    n = the number of columns of matrix A
    i = 0
    rank = 0
    for j = 0; i < n − 1; i + + do
        if i == m then
            Break
        end if
        p = the index of the maximum element in the j-th column from the i-th row to the (m-1)-th row
        if p > 0 then
            Switch the i-th row with the p-th row.
        end if
        if A[i, j]! = 0 then
            rank = rank + 1
            for r = i + 1; r < m − 1; i + +  do
                if A[r, j] == 0 then
                    Continue
                else
                    Subtract the i-th row from the r-th row
                    Take Mod 2 with A
                end if
            end for
            i = i+1
        end if
    end for
end function
```

<table>
<tr><th></th><th>1|23</th><th>23|1</th><th>1|34</th><th>34|1</th><th>2|34</th><th>34|2</th><th>2|41</th><th>41|2</th><th>3|12</th><th>12|3</th><th>3|41</th><th>41|3</th><th>4|12</th><th>12|4</th><th>4|23</th><th>23|4</th></tr>
<tr><td>1|2</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>1|3</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
<tr><td>2|1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td></tr>
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<tr><td>1|4</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td></tr>
<tr><td>4|1</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td></tr>
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<tr><td>4|2</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>1</td><td>0</td></tr>
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<tr><td>4|3</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>0</td><td>1</td><td>0</td><td>0</td><td>1</td><td>0</td></tr>
</table>

Figure 2.2: An example of 2 robots' motion on four edges of a squre

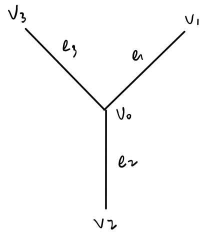


Figure 2.3: Radial 3-Prong Tree

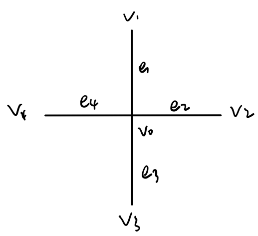


Figure 2.4: Radial 4-Prong Tree

$$\chi(C^n(T_k)) = -(nk - 2n - k + 1)\frac{(n+k-2)!}{(k-1)!} \tag{2.7}$$

, which implies that the configuration space is homotopic (also homologous) to a wedge of $1 - \chi$ cycles.

By computing the cubical homology of discretized configuration spaces, we also hope to get a similar results. In fact, it is. As shown in Table 2.1, we know $\chi(D^2(T_k)) = \beta_0 - \beta_1 + \beta_2 = 1 - \beta_1$, since $\beta_0 = 1$ and $\beta_2 = 0$ for all $3 \leq k \leq 6$. Therefore, the result $\chi(C^2(T_k)) = 1 - \beta_1 = \chi(D^2(T_k))$ shows that the discretization is actually a good method to simplify configuration spaces. Although discretize configuration spaces indeed remove more information from the original spaces, the most important information is still preserved.

2.3.2 n-Polygons

This subsection will show results of two robots on n-polygons, seen in Table 2.2. We see that for every n-polygons, $3 \leq n$, $\beta_0 = 1$ always holds, which means there is only one connected component. Because n-polygons are all connected graph and there is always a path between any two points in the discretized configuration space. $\beta_1 = 1$ means there is only one loop in the quotient space.

Table 2.1: Cubical Betti numbers for Discretized Configuration Spaces of Radial k-Prong Trees

k	**3**	**4**	**5**	**6**
β_0	1	1	1	1
β_1	1	5	11	19
β_2	0	0	0	0
$1 - \chi(C^2(T_k))$	1	5	11	19

From the perspective of trajectories, the only one loop seems trivial, as we see in 2.5 (c), which means two robots rotate the triangle one time. From the perspective of quotient, we can see from Figure 2.6 (c) that, if we hope to transform a loop, starting from 1|3 and ending at 1|3, to any other loop with the same starting and ending point, adding some edges (boundaries of several 2-cubes) is enough.

Furthermore, as we see from Figure 2.5, 2.1 and 2.6, each of their discretized configuration spaces looks like a necklace decorated with some cubes. Actually, all of them can be deformation retracted to a cycle.

Table 2.2: Cubical Betti numbers for Discretized Configuration Spaces of n-polygons

n	**3(triangle)**	**4(square)**	**5**	**6**
β_0	1	1	1	1
β_1	1	1	1	1
β_2	0	0	0	0

2.3.3 Complete Graphs

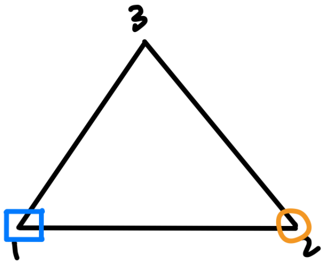
When we add edges between vertices inside n-polygons, this leads us to consider the case of complete graphs. This subsection will show results of two robots on some complete graphs, seen in Table 2.3.

Table 2.3: Cubical Betti numbers for Discretized Configuration Spaces of Complete Graphs

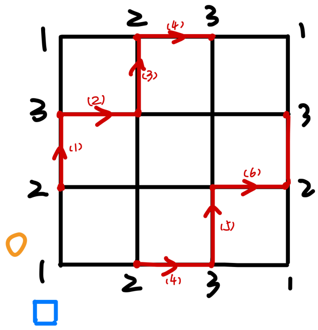
Graphs	$K_{3,3}$	K_4	K_5	K_6
β_1	8	7	12	20
β_2	1	0	1	19

Recall that in the Section 1.3, the second Betti number β_2 counts the number of three-dimensional 'voids' or 'cavities' in a topological space. By cellular homology theory, it is known that cubical and simplicial homology groups are isomorphic, see in [6]. The second cubical Betti number of $D^2(K_{3,3})$ and $D^2(K_5)$ are all one, which means there is only one three-dimensional 'cubical void'. Then it is natural to ask a question that if a discretized configuration space is a manifold or not. As we see from the above examples of n-polygons, they are not manifolds, since they have boundries on which the neighborhood of any point is not homeomorphic to $\mathbb{R}^2$. However, $D^2(K_{3,3})$ and $D^2(K_5)$ are 2-manifolds, as demonstrated in [1].

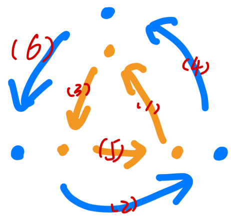
We take the complete graph K_5 for an example. One can draw the discretized configuration space by starting from a vertex (0-cube), i.e., arranging two robots on two distinct points in K_5. Then depict the edges (1-cube) incident on that point by moving the two robots. Finally arrange these edges properly to form possible 2-cubes. Luckily, in $D^2(K_5)$, each edge borders a pair of 2-cubes and the link of each vertex is a closed path. It shows that an neighborhood of every point in $D^2(K_5)$ is homeomorphic to $\mathbb{R}^2$. Besides, consider one 2-cube as two 2-simplices by adding one diagonal line, $D^2(K_5)$ actually admits a triangulation. And the orientation of that is consistent by checking each 2-simplex. Next, since there is a finite number of elementary cubes, $D^2(K_5)$ is compact. Consequently, $D^2(K_5)$ is a closed orientable 2-manifold. By counting the number of



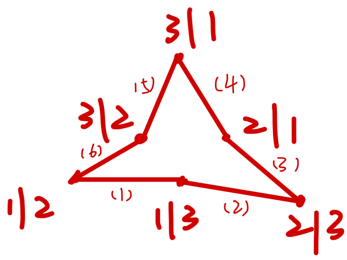
(a) 3-polygon



(b) Discretized Configuration Space in axes



(c) Trajectory



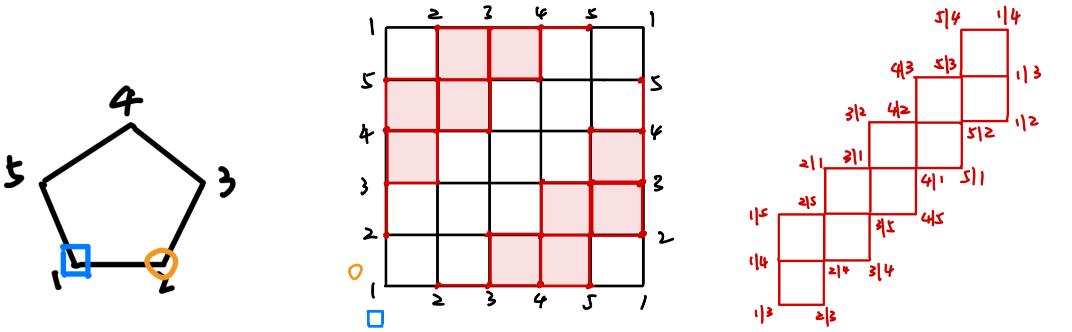
(d) Discretized Configuration Space

Figure 2.5: The discretized configuration space of 2 robots on a 3-polygon in axes

0-cubes (vertics), 1-cubes (edges), 2-cubes (faces), we may get the Euler characteristic, which, by the classification theorem in Section 1.3.2, determines the 2-manifold up to homeomorphism:

$$\begin{aligned} \chi(D^2(K_5)) &= \#\text{vertices} - \#\text{edges} + \#\text{faces} = 20 - 60 + 30 \quad = -10 \\ &= \beta_0 - \beta_1 + \beta_2 = 1 - 12 + 1 = -10 \end{aligned} \tag{2.8}$$

As shown above, it also matches Euler-Poincare Theorem mentioned before. Furthermore, $D^2(K_5)$, as an orientable 2-manifold, has $g = 1 - \frac{1}{2}\chi = 6$ holes, which corresponds to 12 working loops as shown by $\beta_1 = 12$. It is also an example that illustrates Betti number is an refinement of Euler characteristic.



(a) 5-polygon

(b) Discretized Configuration Space in axes

(c) Discretized Configuration Space

Figure 2.6: The discretized configuration space of 2 robots on a 5-polygon

2.4 Future Research

It is interesting to see that the discretized configuration spaces of two robots on $K_{3,3}$ and K_5 are manifolds, despite the fact that $K_{3,3}$ and K_5 are only two non-planar graphs.

We make an reasonable assumption that the discretized configuration spaces of $K_{3,3}$ and K_5 are the only two manifolds. However, it still needs time prove that proposition. It is also worth investigating if there is a deeper principle underlying the relationship between graphs and discretized configuration spaces.

Moreover, with simple calculation by computer, we can get cubical 3rd Betti number β_3 easily. However, it is still not clear about how to interpret the meaning of that directly. Similarly, we are now able to compute the number of motion cycles of 3 or more robots by implementing the above algorithm, but higher order Betti numbers are still mysterious for us.

Chapter 3

Conclusion

The theory of homology as shown in this thesis has been proved to be powerful with respect to the application of robotic motion planning. Discretized configuration spaces can be analyzed via linear algebra. The impossible task, to visualize 2 or 3 or more robots functioning in a simple graph, has been converted into topological questions. However, how to convert those information in configuration spaces back into the reality and interpret them convincingly is still a problem. It seems hard to explain the meaning of the 3rd Betti number for 2 robots' motion.

Despite that, the method provided in the thesis can be utilized to test the accuracy of a theoretical method that can determine the information from configuration spaces. For instance, in Section 2.3.1, we tested the explicit formula for radial k-prong trees in [1].

It is also reasonable to believe that there is a underlying relationship between graphs and configuration spaces. It is amazing to see that non-planar graphs are turned to become manifolds in configuration spaces. Using both algebraic and computational methods might provide more insights into the robotic motion planning question in the future.

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