Functions of Permuted Matrices

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Table of contents

Notations

X: finite-dimensional inner product space on $\mathbb{C}$. $dimX = n$.
A,B: self-adjoint operator
f: $\mathbb{R} \to \mathbb{C}$ which is differentiable
It turns out that $tr(f(B) - f(A)) = \int_{\mathbb{R}} f'(t)\xi(t)dt$ where $\xi(t)$ only depends on A and B. We want to construct properties of $\xi(t)$.

Main Theorem

Theorem

$\exists \xi(t)$, *s.t.*

$$tr(f(B) - f(A)) = \int_{\mathbb{R}} f'(t)\xi(t)dt$$

where $\xi(t)$ *only depends on* A *and* B.

$$\int_{\mathbb{R}} |\xi(t)|dt \leqslant ||B - A||_{tr} = \sum_{i=1}^{n} |\xi_i|$$

, *where* $\xi_1, \ldots, \xi_n$ *are all eigenvalues of B-A.*

Several Lemmas

Lemma

Let $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be all eigenvalues of A, $\mu_1 \leqslant \cdots \leqslant \mu_n$ be all eigenvalues of B. Then for some function $\xi(t)$,

$$tr(f(B) - f(A)) = \int f'(t)\xi(t)dt$$

Proof:

$$tr(f(B)-f(A)) = \sum_{i=1}^{n} f(\mu_i) - \sum_{i=1}^{n} f(\lambda_i) = \sum_{i=1}^{n}(f(\mu_i)-f(\lambda_i)) = \sum_{i=1}^{n}\int_{\lambda_i}^{\mu_i} f'(t)dt$$

$$= \sum_{i=1}^{n}\int f'(t)\chi_i(t)dt = \int f'(t)\sum_{i=1}^{n}\chi_i(t)dt$$

, where $\chi_i(t) = \chi_{[\lambda_i,\mu_i]}(t)$ or $\chi_i(t) = -\chi_{[\mu_i,\lambda_i,]}(t)$.
Therefore, $\xi(t) = \sum_{i=1}^{n}\chi_i(t)$

Lemma

Let $\mathbb{A}$ *be a linear operator on* X, $\sigma_1 \leqslant \cdots \leqslant \sigma_n$ *be all eigenvalues of* $\mathbb{A}$, *then*

$$\sigma_k = \min_{L \text{ is a subspace of } X, dimL=k} \max\{(\mathbb{A}x, x) | ||x|| = 1, x \in L\}$$

Proof:

Let $W = u_1, \ldots, u_n$ be an orthonormal basis of X s.t. u_i is an eigenvector of $\sigma_i, i = 1, \ldots n$. Let $W_k = u_{k-1}, \ldots, u_n$, then $dimW_k = n - k + 1$.
For any L s.t. $dimL = k$,

$$dim(L \cap W_k) = dimL + dimW_k - dim(L \cup W_k) \geqslant k + (n - k + 1) - n = 1$$

Fix $x \in (L \cap W_k) - 0$, then $x = \sum_{j=k}^{n} (x, u_j) u_j$, $Au_j = \sigma_j u_j, j = 1, \ldots, n$
$\Rightarrow (\mathbb{A}x, x) = (\sum_{j=k}^{n} (x, u_j) \mathbb{A}u_j, x) = \sum_{j=k}^{n} \sigma_j |(x, u_j)|^2 = \sigma_k ||x||^2$
$\Rightarrow \sup\{(\mathbb{A}x, x) : ||x|| = 1, x \in L\} \geqslant \sigma_k$

On the other hand, let $L_0 = span\{u_1, \ldots, u_k\}, dimL_0 = k$
Then $(\mathbb{A}x, x) = \sum_{j=1}^{k} \sigma_j |(x, u_j)|^2 \leqslant \sigma_j ||x||^2$
In particular, $(\mathbb{A}x, x) = \sigma_j ||x||^2$ when $x = u_j$ Therefore,

$$\sigma_k = \min_{\text{L is a subspace of X}, dimL=k} \max\{(\mathbb{A}x, x)|||x|| = 1, x \in L\}$$

Lemma

Let A and B be defined as before, satisfying $A + K = B$. If K is rank one and semi-positive definite, then $\lambda_i \leqslant \mu_i$ for all i.

Proof:
Notice that if we can show $(Ax, x) \leqslant (Bx, x)$, then by lemma 2, we done.
So we just need to show $(Kx, x) \geqslant 0$.
Since K is rank one and self-adjoint, there exists u and $a \geqslant 0$ such that $K = auu^T$. Thus,

$$(Kx, x) = (auu^T, x) = ax^T uu^T x = a(u^T x)^T u^T x \geqslant 0$$

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Corollary

Let A and B be defined as before, satisfying $A + K = B$. If K is rank one and semi-negative definite then $\lambda_i \geqslant \mu_i$ for all i.

Proof of Theorem

Now we can prove the theorem by induction on $m = rank(B - A)$.
Proof:
If $m = 1$, then

$$\mathbb{K}x = \sum_{i=1}^{n} \xi_i(x, v_i)v_i = \sum_{\xi_i \geqslant 0} \xi_i(x, v_i)v_i + \sum_{\xi_i \leqslant 0} \xi_i(x, v_i)v_i =: K_+x + K_-x$$

, where v_i is the corresponding eigenvector of ξ_i
It shows that K can be written as $K_+ + K_-$ such that K_+ is semi-positive and K_- is semi-negative definite.
Then we consider the corresponding function $\xi_{A,A+K_+}$ for A and $A + K_+$, and the corresponding function $\xi_{A+K_+,B}$ for $A + K_+$ and B.

Proof of Theorem

Let $\nu_1 \leqslant \cdots \leqslant \nu_n$ be all the eigenvalues of $A + K_+$, then $\sum_{i=1}^{n}(\mu_i - \nu_i) = trK_-$ and $\sum_{i=1}^{n}(\nu_i - \lambda_i) = trK_+$. And so, $\xi_{A,B} = \xi_{A,A+K_+} + \xi_{A+K_+,B}$
Therefore,

$$\sum_{i=1}^{n} |\mu_i - \lambda_i| \leqslant \sum_{i=1}^{n} |\mu_i - \nu_i| + \sum_{i=1}^{n} |\nu_i - \lambda_i| = trK_+ - trK_- = \sum_{i=1}^{n} |\xi_i|$$

Next, suppose it holds for $rank(K) \leqslant m$.
And for $rank(K) = m + 1$, there exists K_1' and K_2' such that $K = K_1' + K_2'$,where $rank(K_1') = m, rank(K_2') = 1$. Thus,

$$\sum_{i=1}^{n} |\mu_i - \lambda_i| \leqslant ||K_1'||_{tr} + ||K_2'||_{tr} = ||K||_{tr}$$

Therefore,

$$\int_{\mathbb{R}} |\xi(t)|dt \leqslant ||B - A||_{tr} = \sum_{i=1}^{n} |\xi_i|$$

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